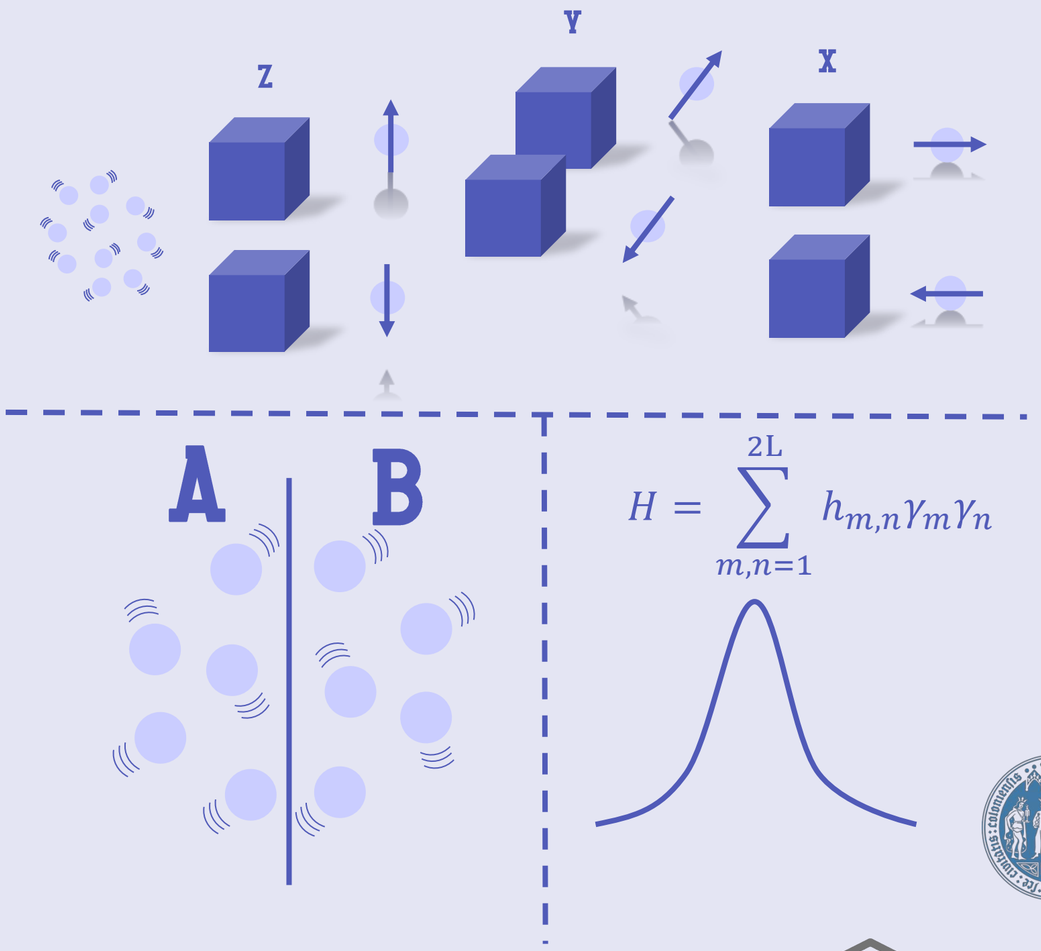
FERMIONIC

NON-LOCAL MAGIC

(and some personal considerations)



Daniele Iannotti, Scuola Superiore Meridionale

Istituto Nazionale di Fisica Nucleare

Scuola Superiore Meridionale

EUROPEAN COOPERATION IN SCIENCE & TECHNOLOGY

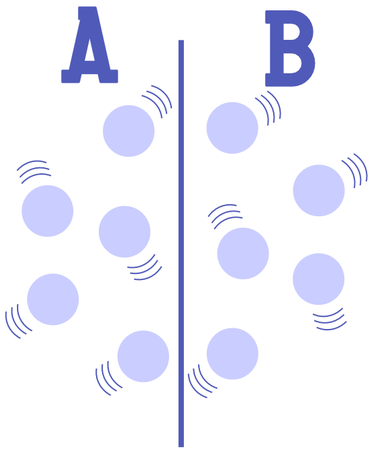
QUANTUM INFORMATION THEORY LAB

QUANTUM RESOURCES FOR QUANTUM COMPUTING

ENTANGLEMENT

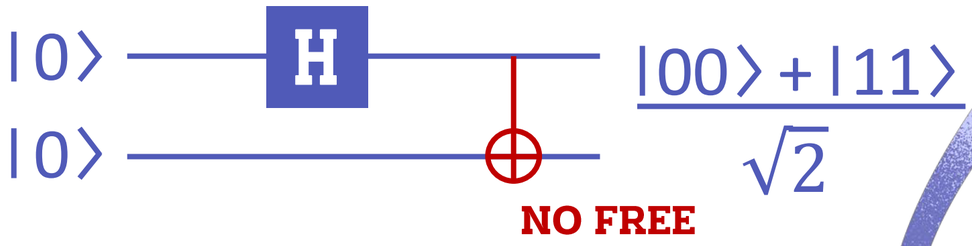
MEANING

Non-classical correlations across subsystems



FREE OPERATIONS

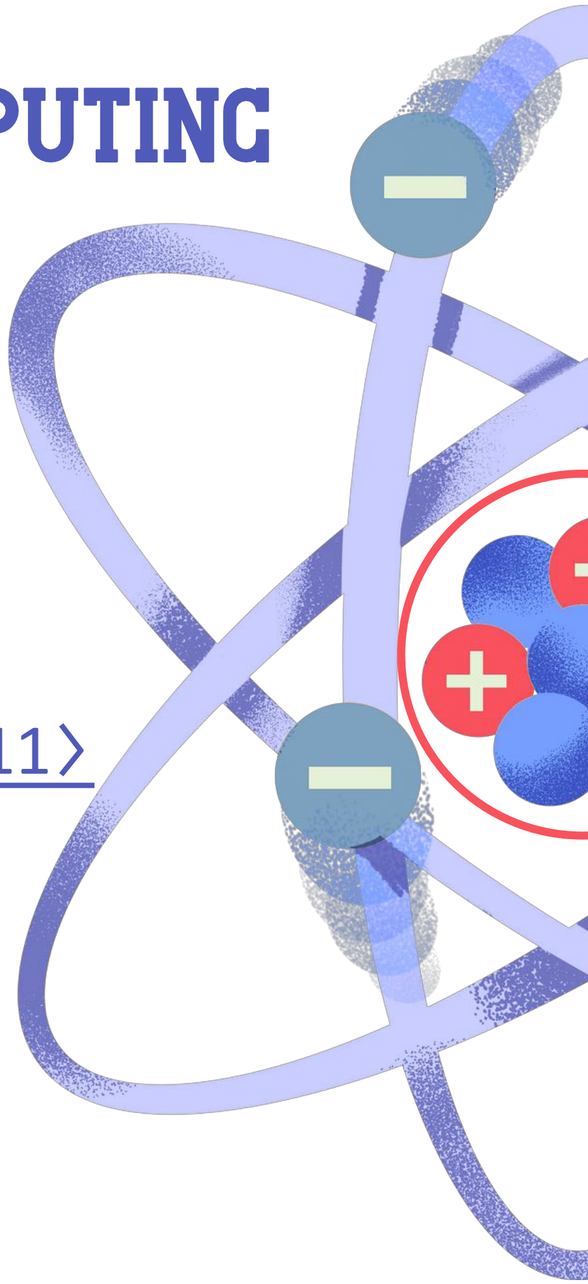
Local Operations and Classical communications



EXPERIMENTALLY MEASURABLE QUANTITY

Subsystem purity

$$\mathrm{E}_{lin}(\psi) = 1 - Tr(\psi_A^2)$$



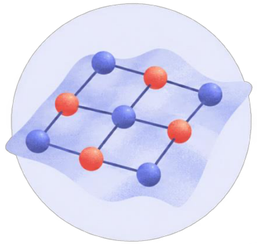
R. Horodecki, et al. "Quantum entanglement." Reviews of modern physics 81.2 (2009): 865-942.



WHAT IS ENTANGLEMENT USEFUL FOR?

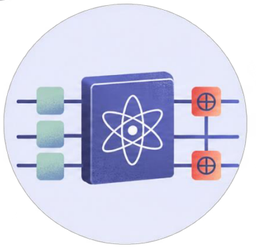
- **Quantum phases**

Vidal et al., PRL 90, 227902 (2003);
Kitaev & Preskill, PRL 96, 110404 (2006);
Levin & Wen, PRL 96, 110405 (2006)...



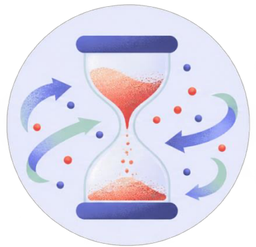
- **Quantum computation**

Jozsa & Linden, Proc. R. Soc. A 459, 2011 (2003);
Vidal, PRL 91, 147902 (2003)...



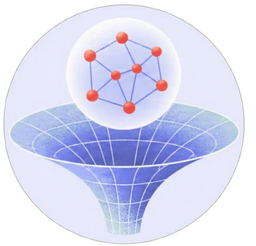
- **Thermalization**

Calabrese & Cardy, J. Stat. Mech. P04010 (2005);
Page, PRL 71, 1291 (1993)...

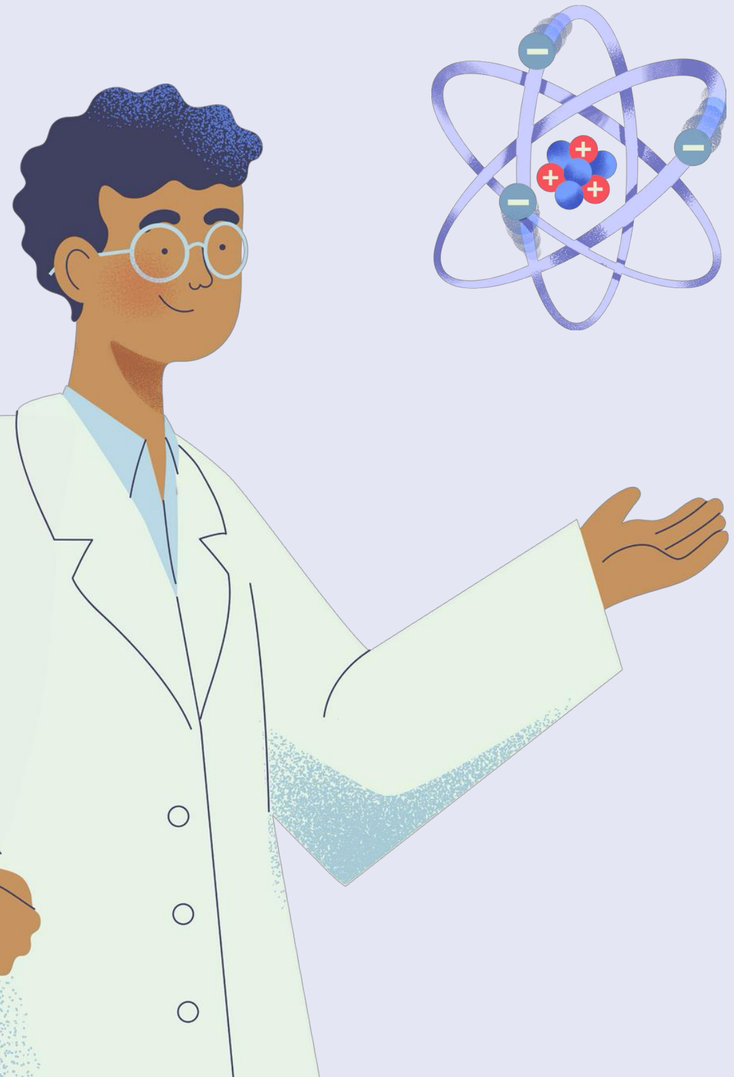


- **Quantum gravity**

Ryu & Takayanagi, PRL 96, 181602 (2006);
Pastawski, Yoshida, Harlow & Preskill, 2015(6), 149 (2015)....



AND MANY MORE



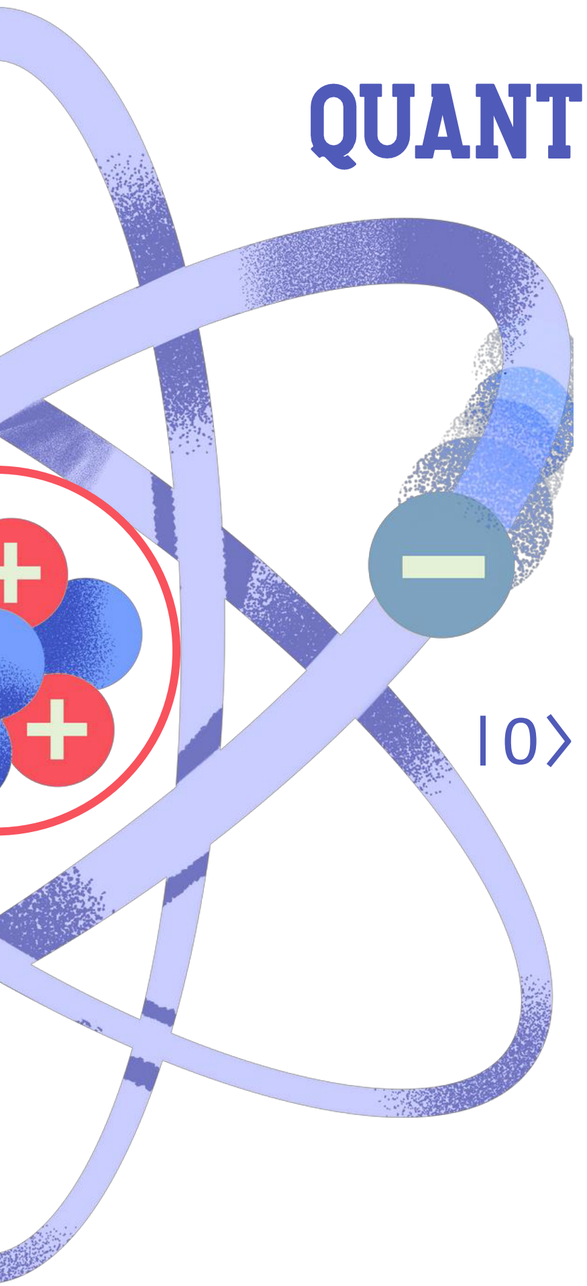
IS ENTANGLEMENT ENOUGH?

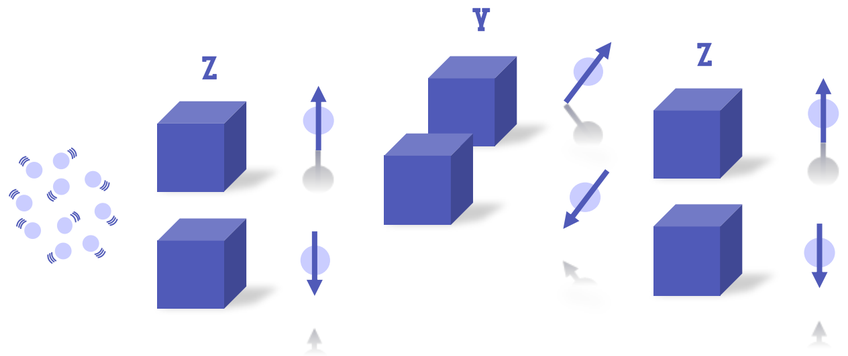
Theorem: A quantum circuit using only the following elements can be simulated efficiently on a classical computer:

- **Preparation of qubits in *computational-basis states, e.g.*** $|0000 \ldots 0\rangle$
- ***Clifford gates, i.e.*** $C\,P\,C^{\dagger} = P'$
- ***Measurements in the computational basis***

Aaronson, S., & Gottesman, D. (2004). Improved simulation of stabilizer circuits. Physical Review A, 70(5), 052328.

QUANTUM RESOURCES FOR QUANTUM COMPUTING





$$|0\rangle \text{ — } \boxed{H} \text{ — } \boxed{T} \text{ — } \frac{|0\rangle + e^{i\frac{\pi}{4}}|1\rangle}{\sqrt{2}}$$

NO FREE

$$M_{lin}(\psi) = 1 - \frac{1}{d}\sum_{P} |\langle\psi|\mathrm{P}|\psi\rangle|^4$$

MEANING

Partial incompatibility

FREE OPERATIONS

Stabilizer operations

EXPERIMENTALLY MEASURABLE QUANTITY

Stabilizer purity

NON-STABILIZERNESS

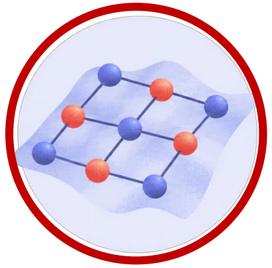
H. Zhu, R. Kueng, M. Grassl& D. Gross. “The Clifford group fails gracefully to be a unitary 4-design.” arXiv:1609.08172 (2016).

L. Bittel, and L. Leone. "Operational interpretation of the Stabilizer Entropy.” Quantum 10, 2069 (2026)

Daniele Iannotti, L. Campos Venuti and A. Hamma. "Van Hove singularities in stabilizer entropy densities." J. Ph. A: Mat. and Th., 2026

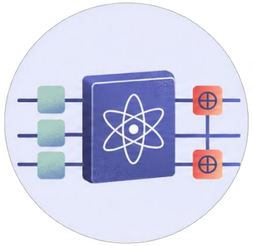
WHAT DOES MAGIC TELL US ABOUT?

- **Quantum phases**



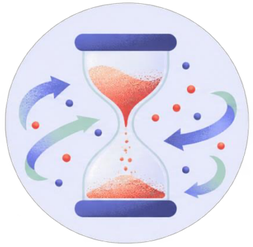
Liu & Winter, PRX Quantum 3, 020333 (2022);
Hoshino, Oshikawa & Ashida, PRX 16, 011037 (2026);
Hoshino & Ashida, PRL 136, 080402 (2026);
Zhang, Kim, Bao, & Vijay (2026) arXiv:2605.15150....

- **Quantum computation**



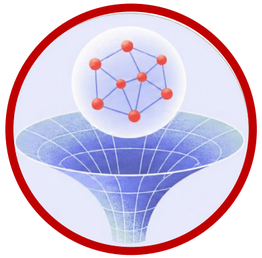
Lami, Haug & De Nardis, PRX Quantum 6, 010345 (2025);
Qian, Huang & Qin, PRL 133, 190402 (2024)...

- **Thermalization**



Leone, Oliviero, Zhou & Hamma, Quantum 5, 453 (2021);
Dowling, Kos & Turkeshi, PRL 135, 050401 (2025)...

- **Quantum gravity**



Cao et al., PRX Quantum 6, 040375 (2025);
Cao et al. (2026), arXiv:2603.13475;
Biswas et al. (2026), arXiv:2607.12047...

AND MANY MORE



NON-STABILIZERNESS

Stabilizer/Little Group:

$$G_\phi = \{g \in G \mid g \cdot \phi = \phi\} \subset G$$

Pauli Group:

$$\mathbb{P}_L = \{\pm 1, \pm i\} \times \{\mathbb{I}, X, Y, Z\}^{\otimes L}$$

Clifford Group:

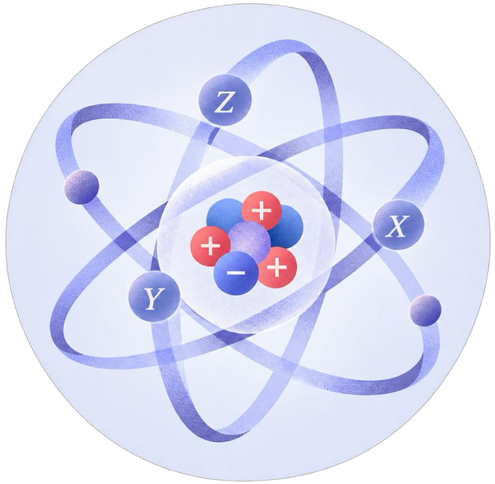
$$\mathfrak{C} = \{ C \subset U(L), P, P' \subset \mathbb{P}_L \mid CPC^\dagger = P' \}$$

Pure stabilizer states

$$\mathrm{PSTAB} = \{|\phi\rangle\langle\phi| = \frac{1}{|G_\phi|} \sum_{P \in G_\phi} P \;|G_\phi \subset \mathbb{P}_L\}$$

$$= \{C|0\rangle^{\otimes L} \mid C \subset \mathfrak{C}\}$$





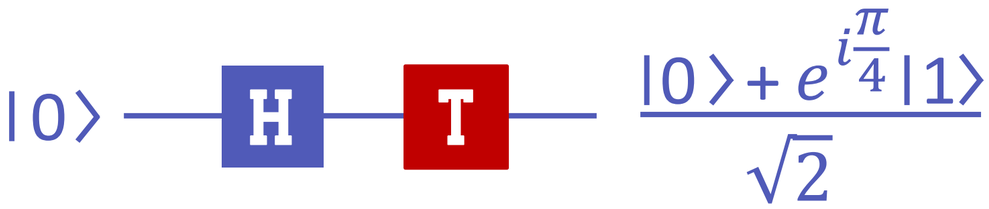
STABILIZER RÉNYI ENTROPIES

$$M_\alpha(|\psi\rangle) = \frac{1}{1-\alpha}\log \Xi_\alpha(|\psi\rangle)$$

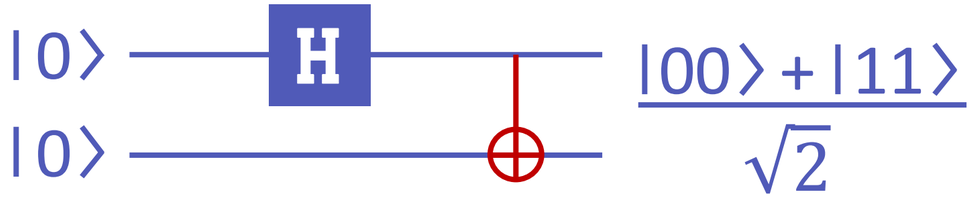
$$\Xi_\alpha(|\psi\rangle) = \frac{1}{2^L}\sum_{P\in\mathbb{P}_L} |\langle\psi|P|\psi\rangle|^{2\alpha}$$

- **FAITHFULNESS** $M_\alpha(|\psi\rangle) = 0 \iff |\psi\rangle \in \mathrm{PSTAB}$, otherwise $M_\alpha(|\psi\rangle) > 0$
- **STABILITY** $M_\alpha(C|\psi\rangle) = M_\alpha(|\psi\rangle)$ with C Clifford operations
- **ADDITIVITY** $M_\alpha(|\psi\rangle\otimes|\phi\rangle) = M_\alpha(|\psi\rangle) + M_\alpha(|\phi\rangle)$

Magic state

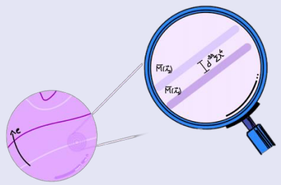


Stabilizer state



Leone, L., Oliviero, S. F., & Hamma, A. (2022). Stabiliser Rényi entropy. Physical Review Letters, 128(5), 050402.

Leone, L., & Bittel, L. (2024). Stabilizer entropies are monotones for magic-state resource theory. Physical Review A, 110(4), L040403.

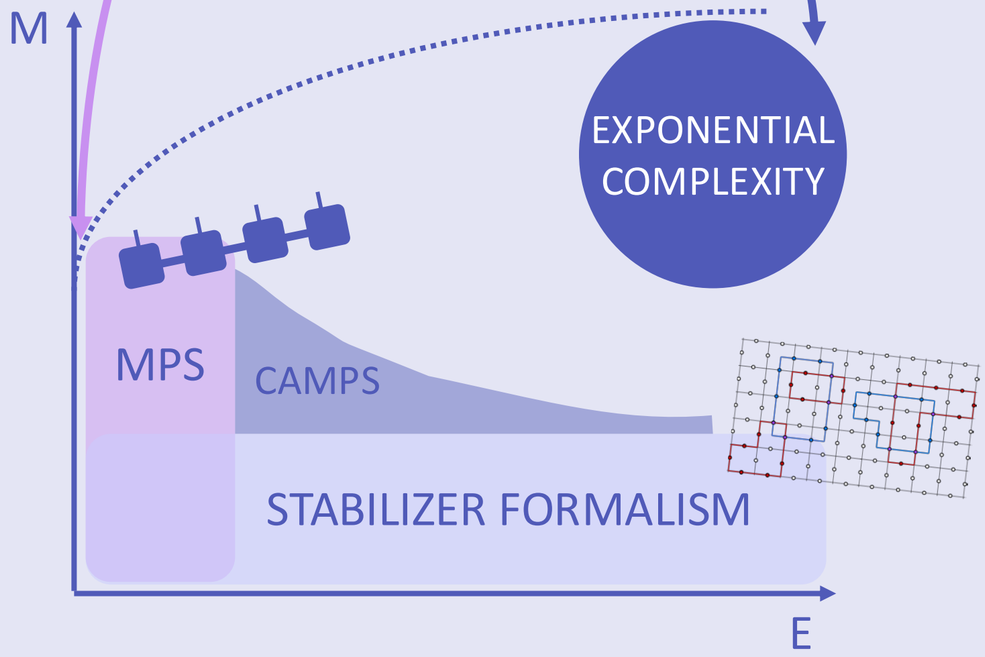


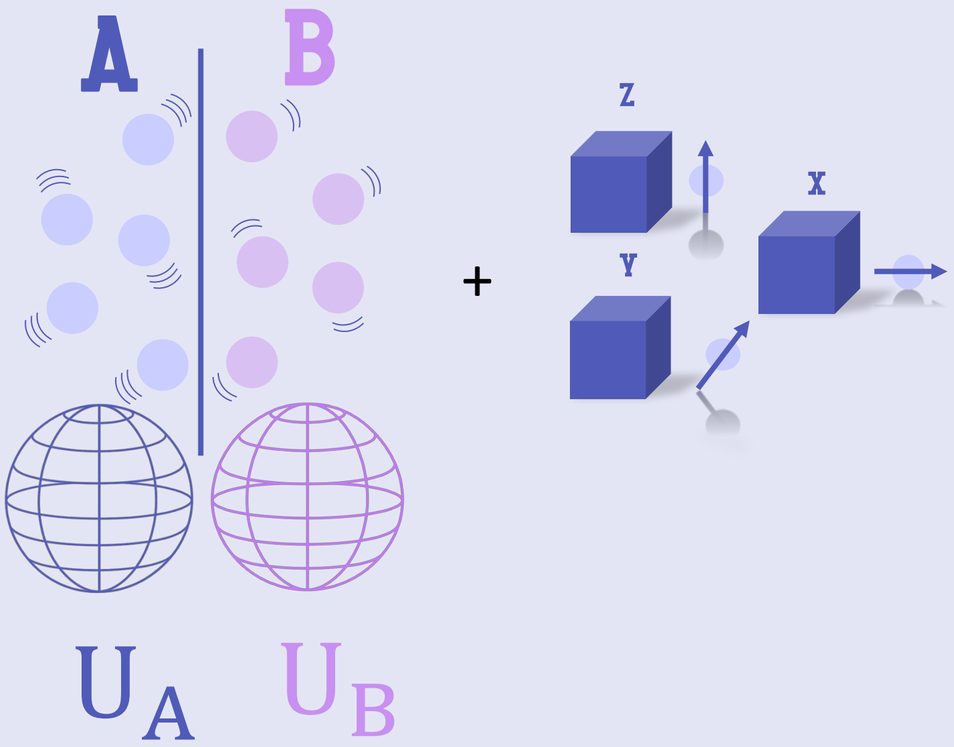
INTERPLAY BETWEEN MAGIC AND ENTANGLEMENT

Entanglement makes room for magic:

The separable state with maximal magic on a single qubit is much less resourceful than its average

$$\mathrm{M}\left(\mathrm{G}^{\otimes \mathrm{L}}\right) \ll \mathbb{E}[\mathrm{M}(|\psi\rangle)]$$





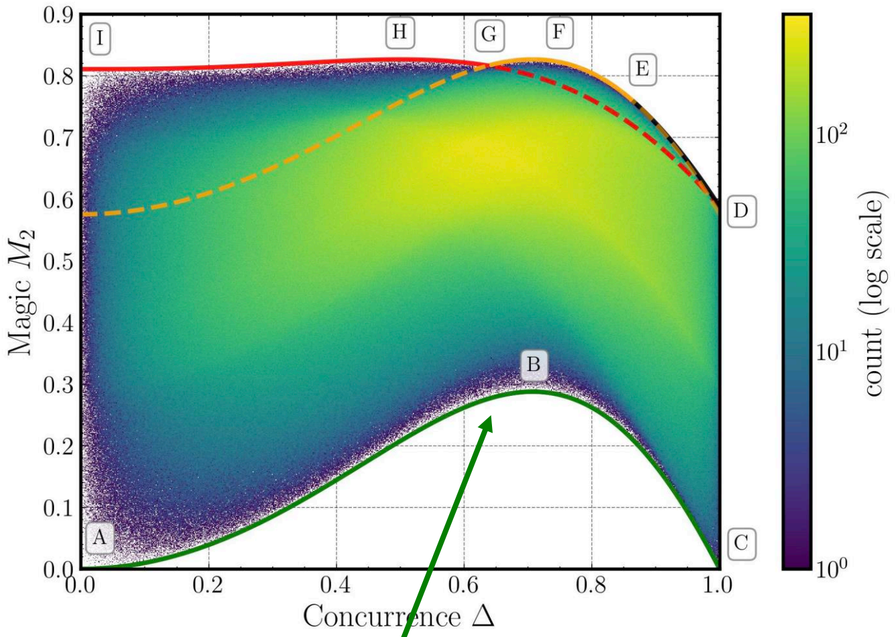
The amount of magic that is left in the correlations* of the system

$$\mathcal{M}^{\mathrm{NL}}(|\psi\rangle) = \min_{\mathrm{U_A}\otimes\mathrm{U_B}} \mathrm{M}(\mathrm{U_A}\otimes \mathrm{U_B}|\psi\rangle)$$

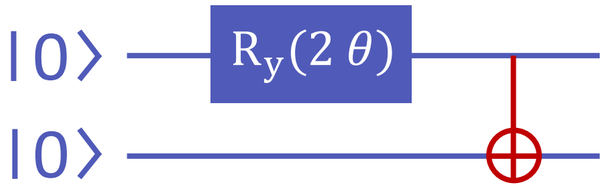
C. ChunJun, G. Cheng, A. Hamma, L. Leone, W. Munizzi, and S. FE Oliviero. "Gravitational backreaction is magical." PRX Quantum 6, no. 4 (2025): 040375

* F. Wei, R. Wang, Y. Zhang, H. Li, J. Li, J., F. Yan, & Z. W. Liu (2026). "Entirely nonlocal quantum magic without entanglement". arXiv :2608.25950

2-QUBIT EXAMPLE



Non-local magic



$\cos(\theta)\,|00\rangle + \sin(\theta)\,|11\rangle$

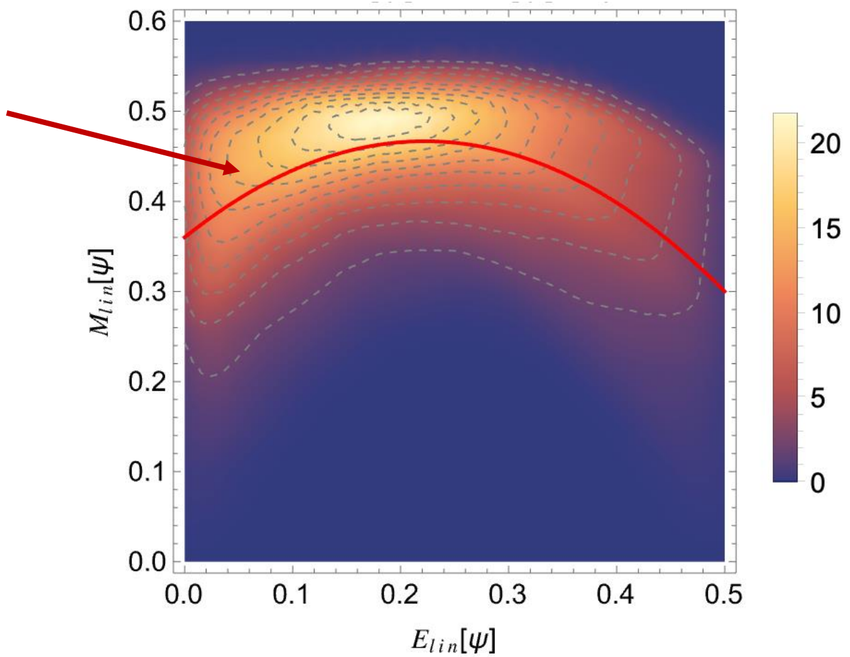
UNIFORM DISTRIBUTION

$$|\psi\rangle = U\,|0\rangle, U \sim Haar\big(U(d)\big)$$

$$\mathbb{E}[f(\psi)] = \int f(U|0\rangle)\,dU$$

Averaged magic
over the Schmidt orbit

$$\mathop{\mathbb{E}}_{U_A \otimes U_B} M(U_A \otimes U_B|\psi\rangle)$$



Bengtsson and K. Zyczkowski, Geometry of Quantum States: An Introduction to Quantum Entanglement. Cambridge University Press, 2006.
Iannotti, D., Esposito, G., Venuti, L. C., & Hamma, A. (2025). Entanglement and stabilizer entropies of random bipartite pure quantum states. Quantum, 9, 1797.
Roman, A.,et al. (2026). The Pareto Frontiers of Magic and Entanglement: The Case of Two Qubits. arXiv:2603.24902.

NON–LOCAL MAGIC & FREE FERMIONS

Free Fermionic Hamiltonian systems

$$H = \sum_{m,n=1}^{2L} h_{m,n}\gamma_m\gamma_n$$



Majorana operators

$$\{\gamma_m, \gamma_n\} = 2\,\delta_{mn}\mathbb{I}$$
$$\gamma_m = \gamma_m^\dagger$$

Covariance matrix

$$\Gamma_{mn} = \frac{i}{2}\langle\psi|[\gamma_m, \gamma_n]|\psi\rangle$$

Fermionic Gaussian state $|\psi\rangle\langle\psi| = e^{\frac{i}{2}\sum_{m,n}\Gamma_{m,n}\gamma_m\gamma_n}$

For fermionic Gaussian states one have that

$$M_2^{\mathrm{FNL}}(|\psi\rangle) = \sum_{i=1}^{\ell} m_2(\lambda_i^2)$$

with $m_2(x) = -\log(1 - x + x^2)$ **and** λ_i **are the positive eigenvalues of the reduced covariance matrix** Γ_A

Daniele Iannotti, B. Magni, R. Cioli, A. Hamma and X. Turkeshi " Nonlocal magic resources of fermionic Gaussian states" arXiv:2604.27049
M. Collura, B. Béri, E. Tirrito" Nonlocal nonstabilizerness in free fermion models" arXiv:2604.27055

INSTRUCTIVE EXAMPLE

$$|\psi_{rb}\rangle = \bigotimes_{j=1}^{\ell} \frac{1}{\sqrt{2}} (|0_j\, 0_{\bar{j}}\rangle + |1_j\, 1_{\bar{j}}\rangle)$$

Note:

Ground state of a free-fermion hopping Hamiltonian with suitable long-range pairing

Maximal entanglement entropy → **Yet a stabilizer state**

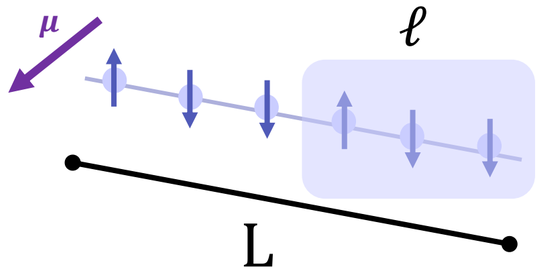
$S(|\psi_{rb}\rangle) = \ell$ **bits**

Only introducing magic non-locally between the partitions will result in non-trivial complex states



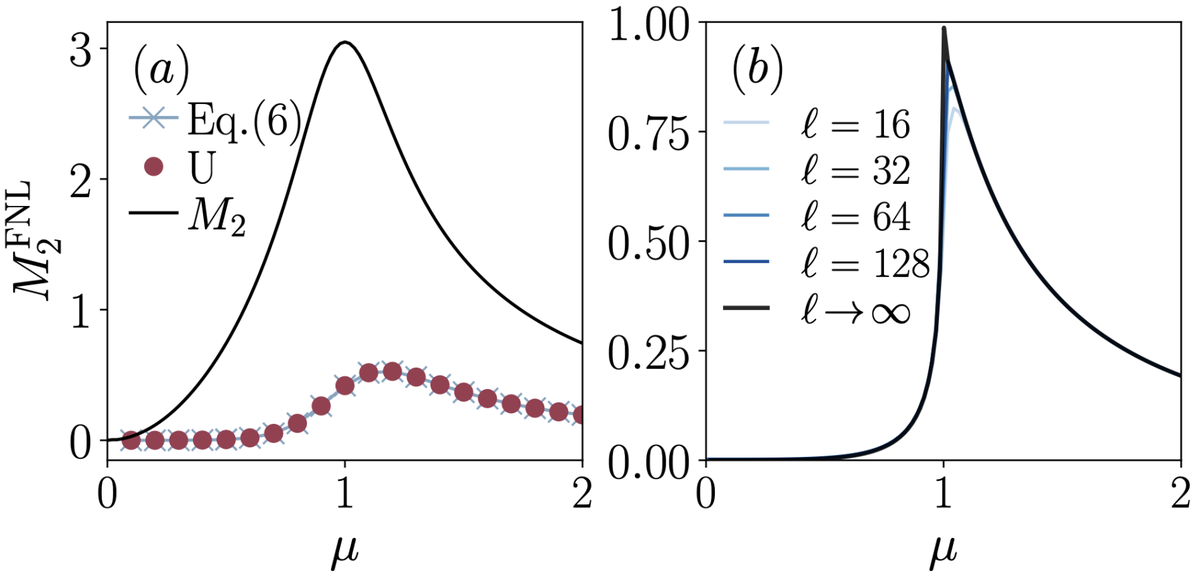
1D

Transverse field Ising model



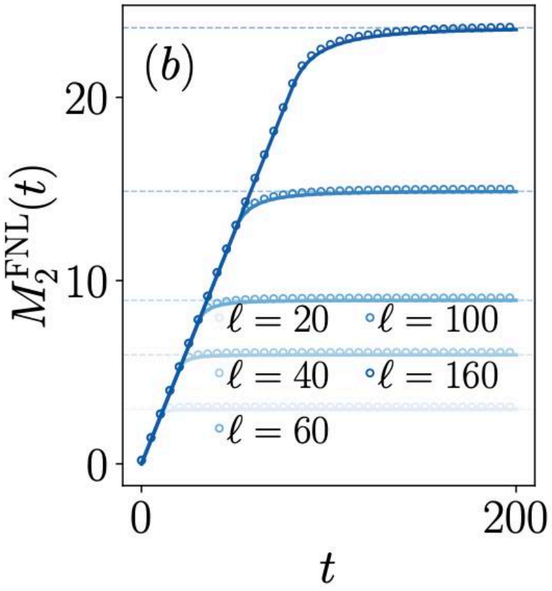
$$H_{TFIM} = \sum_{j=1}^{L} (Z_j Z_{j+1} + \mu X_j)$$

EQUILIBRIUM



$$M_2^{\mathrm{FNL}}(\ell) \simeq \beta_2^{\mathrm{FNL}} \log \ell$$

OUT–OF–EQUILIBRIUM



Non-local magic spreads through ballistic propagation of entangled quasiparticle pairs created by the quench

$$M_2^{\mathrm{FNL}}(\ell, t) \simeq \int_0^{\pi} \frac{dk}{\pi} \min(2|v_k|t, \ell)\, m_2(\sin^2 \Delta\vartheta_k)$$

Daniele Iannotti. , B. Magni, R. Cioli, A. Hamma and X. Turkeshi " Nonlocal magic resources of fermionic Gaussian states" arXiv:2604.27049

FERMIONIC RANDOM GAUSSIAN STATES

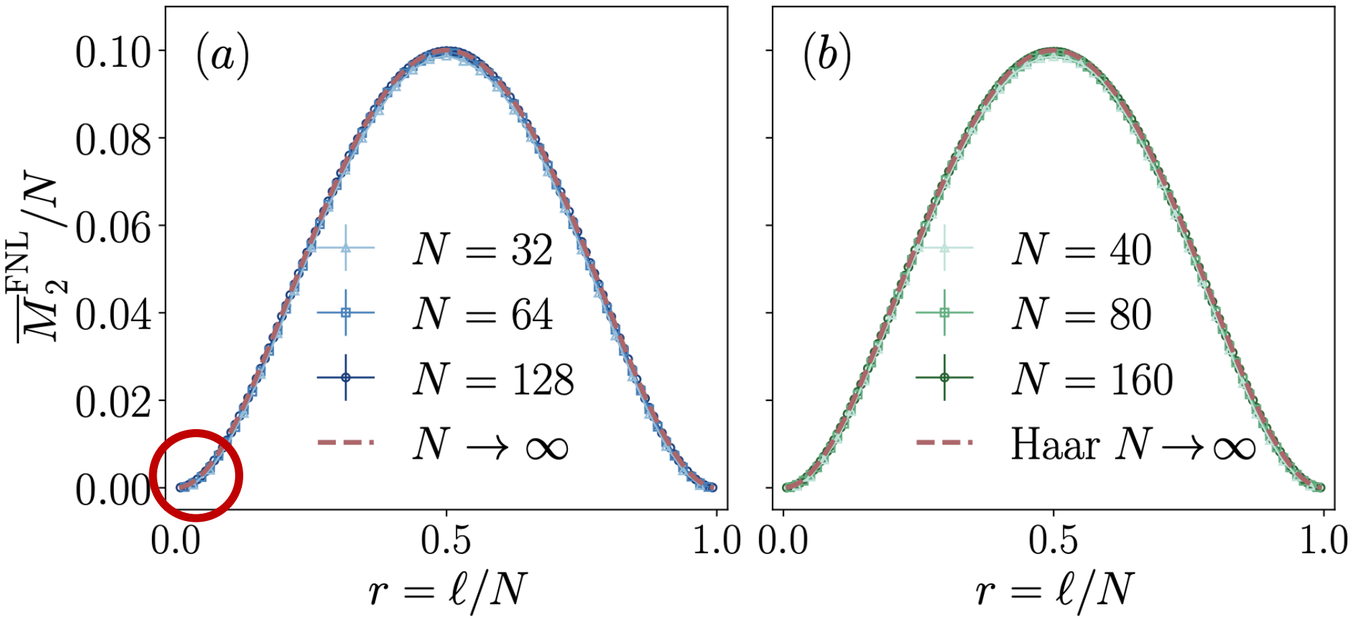
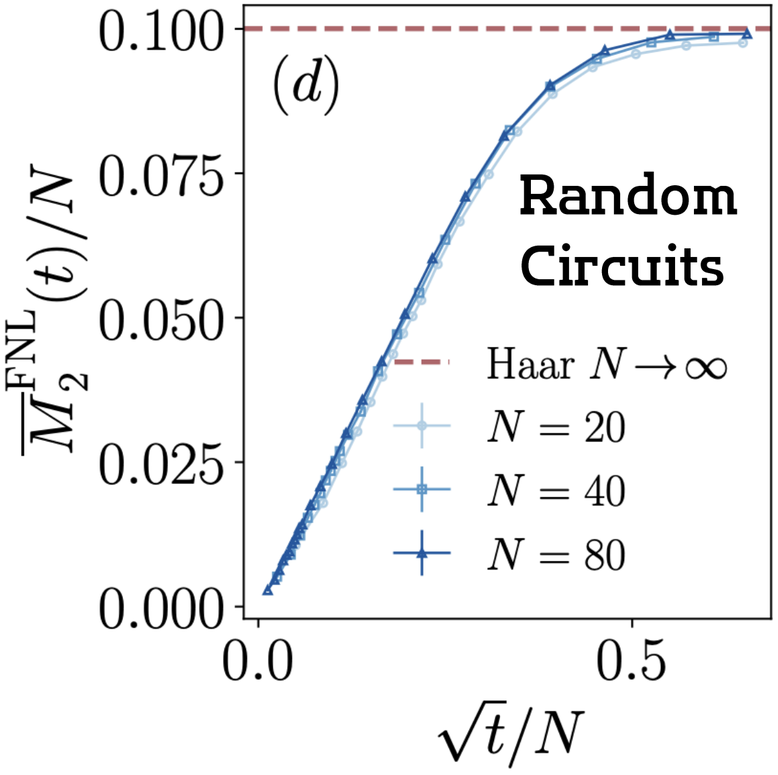


FIG. 1. **Fermionic non-local magic of typical fermionic Gaussian states.** (a) Page curve for Haar-random FGS: $\overline{M_2}^{\mathrm{FNL}}/N$ is plotted against the ratio $r = \ell/N$. The results are compared with the closed form expression for $N \to \infty$ in Eq. (18). (b) The fermionic non-local magic Page curve is tested against the numerically computed average non-local magic for SYK_2 mid-spectrum eigenstates. The results are in quantitative agreement with Eq. (18).

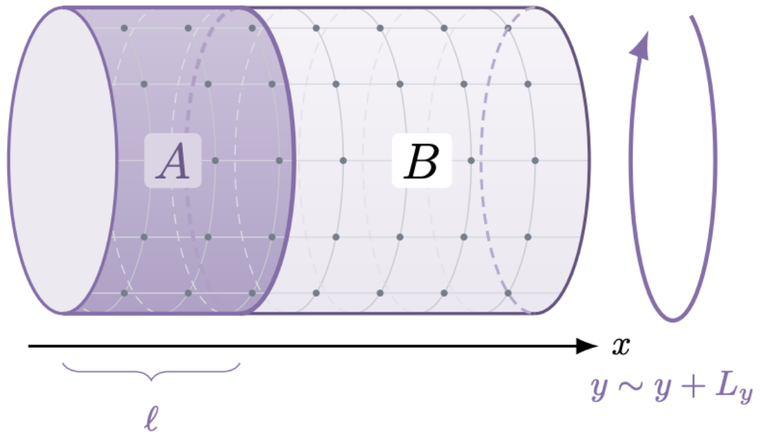


Daniele Iannotti. , B. Magni, R. Cioli, A. Hamma and X. Turkeshi " Nonlocal magic resources of fermionic Gaussian states" arXiv:2604.27049

2D

HIGHER DIMENSIONAL SYSTEMS

(a) Strip bipartition on a cylinder



Fourier transform along y

(b) Moment decomposition

$$\Gamma_A \simeq \Gamma_A(k_1) \oplus \Gamma_A(k_2) \oplus \cdots \oplus \Gamma_A(k_{L_y})$$

k_1 k_2 k_{L_y}

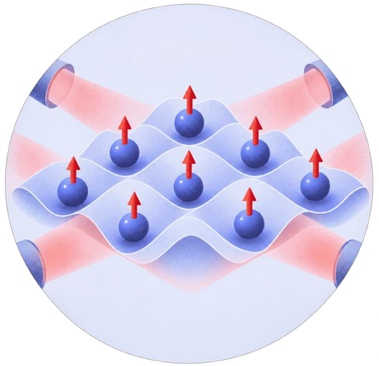
$$M_\alpha^{\mathrm{FNL}}(A) = \sum_k M_{\alpha,k}^{\mathrm{FNL}}, \quad M_{\alpha,k}^{\mathrm{FNL}} = \sum_{j=1}^{\ell} m_\alpha\left(\lambda_j^2(k)\right)$$

Magni, B., Iannotti, D., & Travaglino, R. “Entanglement spectrum and magic in higher-dimensional free fermionic systems”. arXiv:2609.36008.

2D

Nearest Neighbours Hopping model

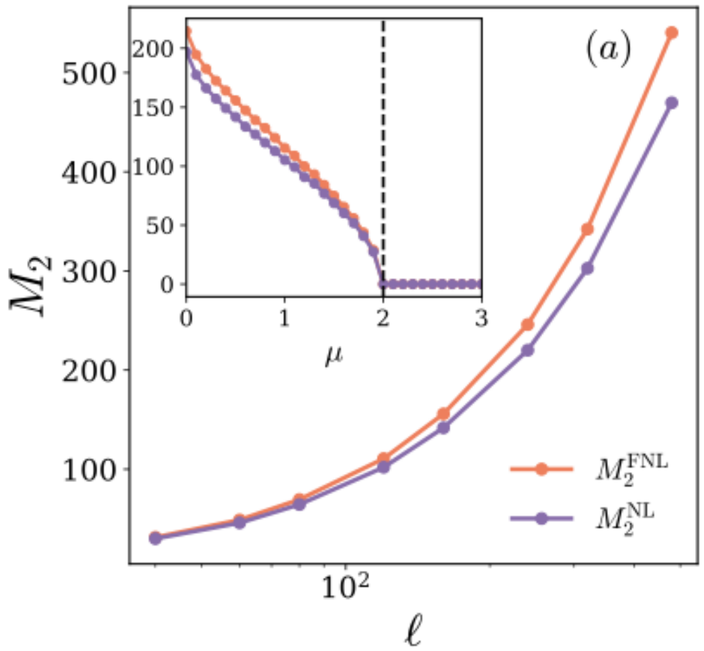
$$H = -\frac{1}{2}\sum_{\langle r\, r'\rangle}^{L}\left(c_r^\dagger c_{r'} + c_{r'}^\dagger c_r\right) + \mu \sum_r c_r^\dagger c_r$$

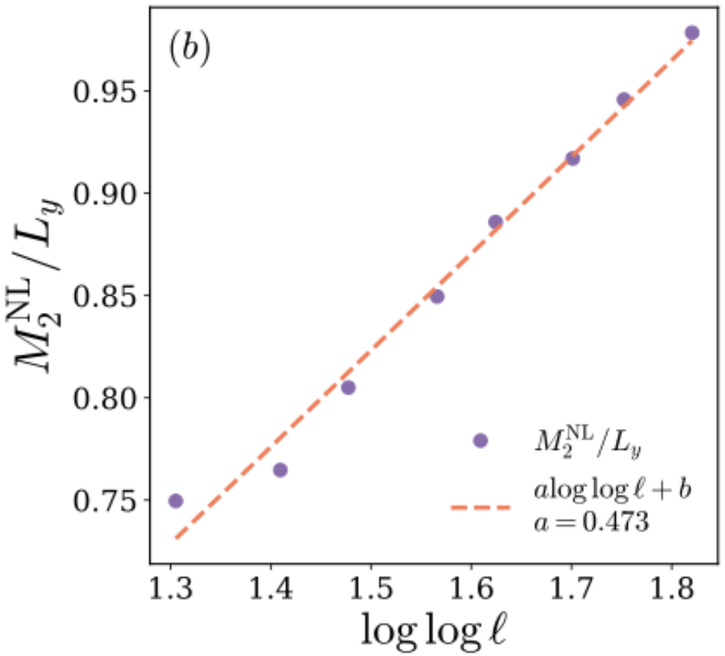


EQUILIBRIUM

Fracrtion of critical modes

$$M_2^{\mathrm{FNL}}(\ell) \simeq 2\, L_y\, f_{L_y}(\mu)\left[\beta_2^{\mathrm{FNL}}\left(\log(2\ell) + 1 + \gamma_E\right) - J_2/\pi^2\right] + \beta_2^{\mathrm{FNL}} A_{L_y}(\mu)$$





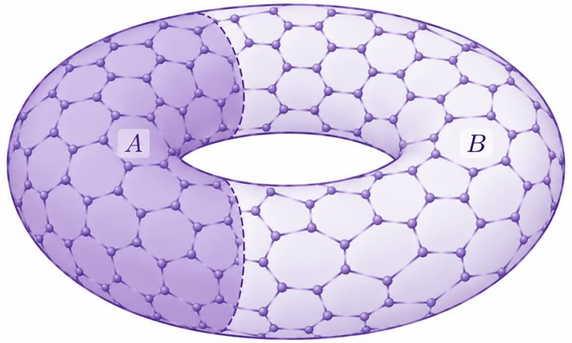
Magni, B., Iannotti, D., & Travaglino, R. “Entanglement spectrum and magic in higher-dimensional free fermionic systems”. arXiv:2609.36008.

2D KITAEV MODEL

$$H = -J_x \sum_{\langle i,j\rangle_x} \sigma_i^x \sigma_j^x - J_y \sum_{\langle i,j\rangle_y} \sigma_i^y \sigma_j^y - J_z \sum_{\langle i,j\rangle_z} \sigma_i^z \sigma_j^z$$

Gapped ground state

$$- J' \sum_{\langle ijk\rangle_{\alpha\beta\gamma}} \sigma_i^\alpha \sigma_j^\beta \sigma_k^\gamma$$

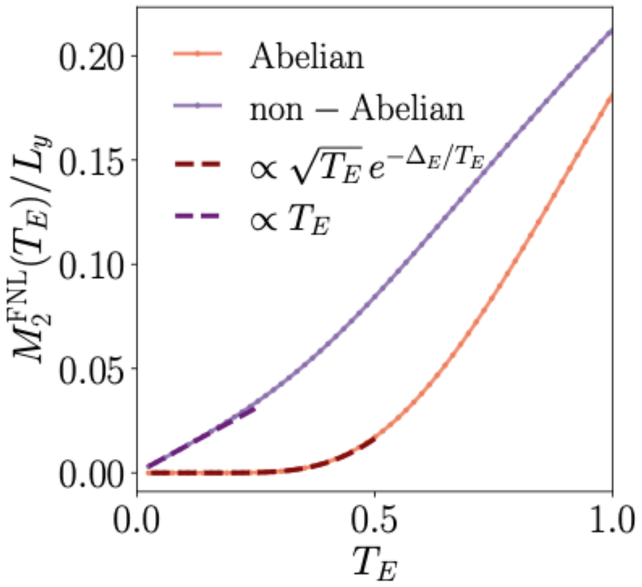


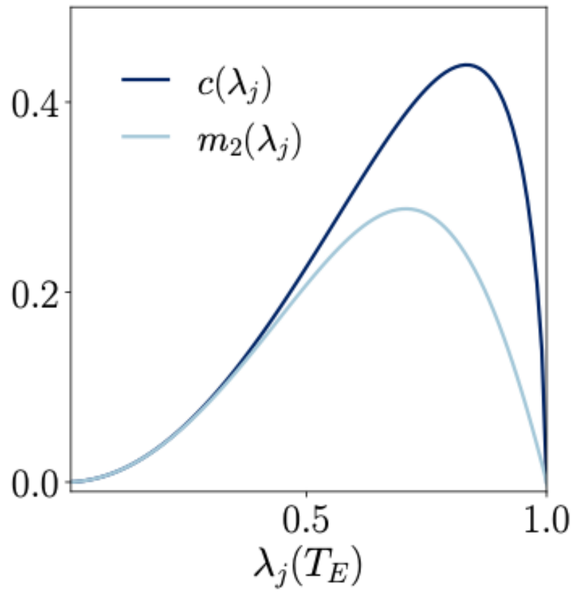
When mapped to fermions is equivalent to a free Fermions coupled to a static $\mathbb{Z}_2$ Gauge field

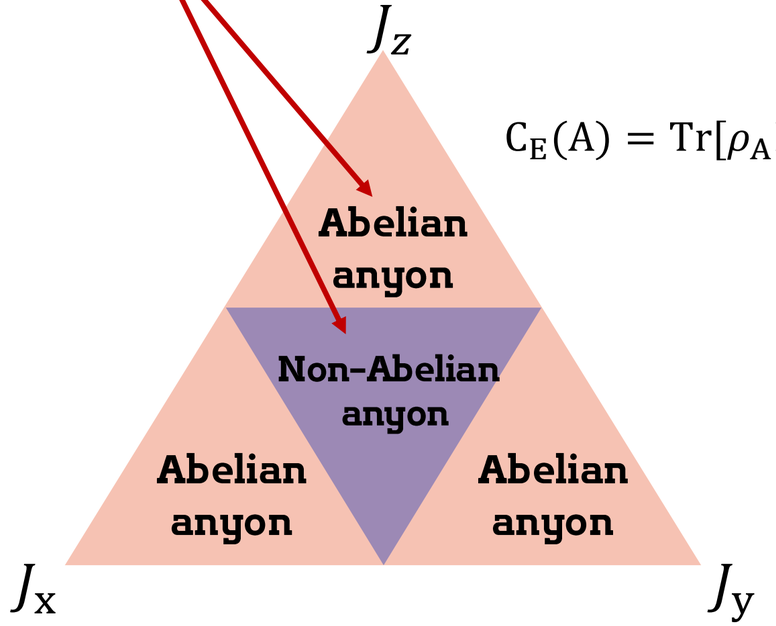
$$\rho_A = \frac{e^{-K_A/T_E}}{\mathrm{Tr}[e^{-K_A/T_E}]}; \qquad K_A = -\log \rho_A$$

$$S(A) = \mathrm{Tr}[\rho_A K_A] = S_F(A) + S_G(A)$$

Topological entanglement entropy



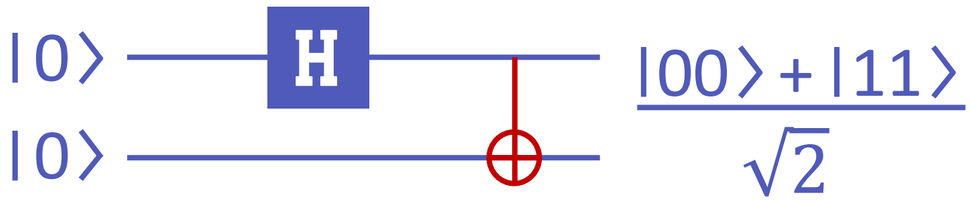




$$C_E(A) = \mathrm{Tr}[\rho_A K_A{}^2] - (\mathrm{Tr}[\rho_A K_A])^2$$

Magni, B., Iannotti, D., & Travaglino, R. "Entanglement spectrum and magic in higher-dimensional free fermionic systems". arXiv:2609.36008.
Lamma, E., Bauer, T., Dalmonte, M., & Collura, M. (2026). Magic of Kitaev spin liquids. arXiv preprint arXiv:2608.25929. Kitaev, A. (2006).

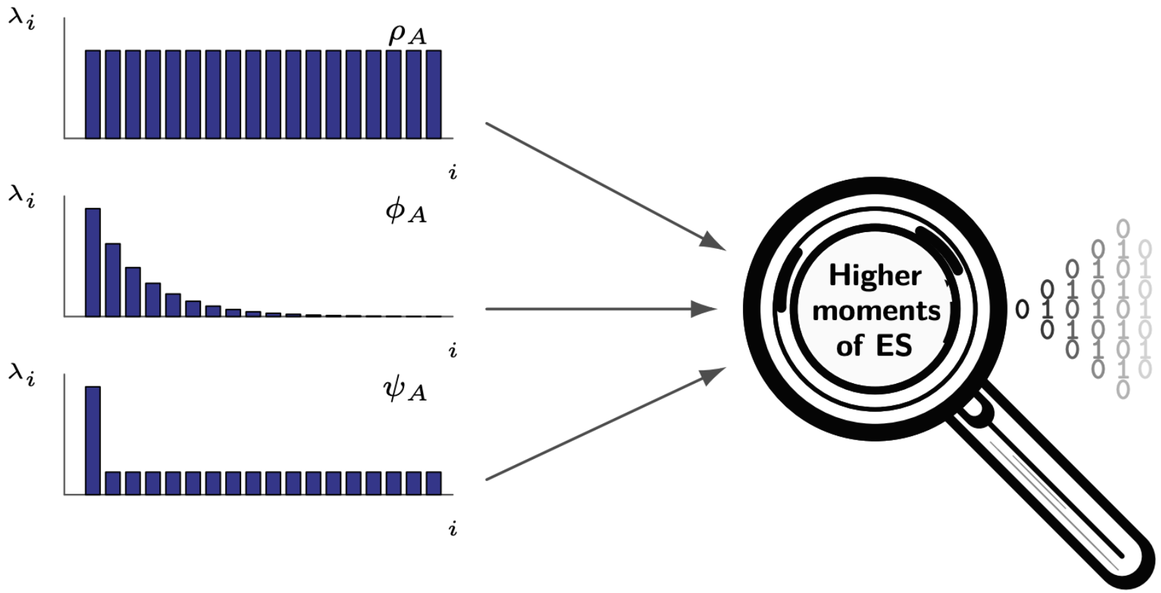
ANTIFLATNESS



Note:

The reduced density matrix, $\rho_A = \frac{\mathbb{I}}{2}$, has a flat spectrum

Entanglement spectrum

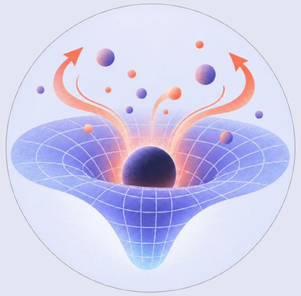



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INFORMATION FROM SECOND-ORDER MOMENT

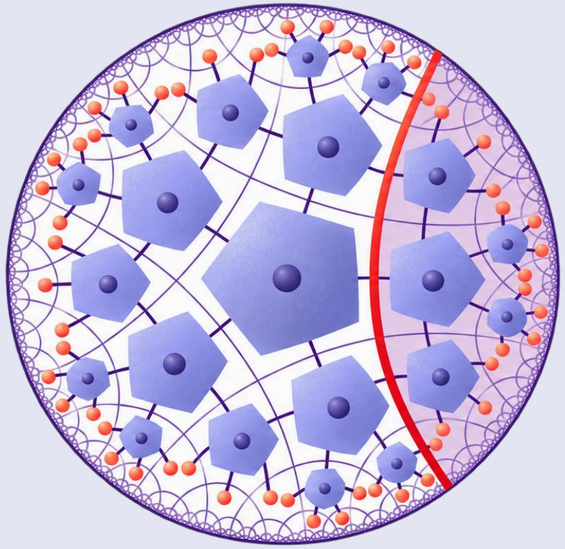
> interplay entanglement and magic
> state convertibility
> information geometry
> maximal antiflatness
> typicality
> ...■
```



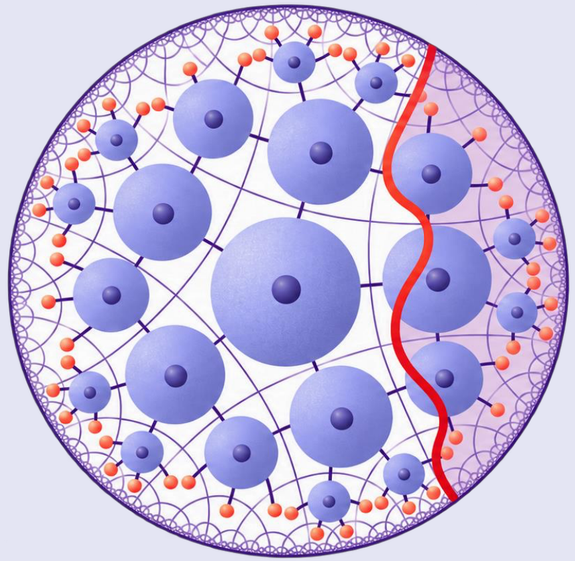
Jasser, B., Iannotti, D., & Hamma, A. (2026). A journey through Flatland: What does the antiflatness of a spectrum teach us?. arXiv:2605.21664.



GRAVITATIONAL BACKREACTION AND MAGIC



HaPPY code



CLiCK-P code

Gravitational backreaction:

The susceptibility of a backreacted surface area $\mathcal{A}$ with respect to the insertion of a cosmic brane with tension $\mathcal{T}$

$$\frac{\partial \mathcal{A}}{\partial \mathcal{T}} = -\, C_E(A)$$

and is connected to the non-local magic of the state

$$\mathcal{M}^{\mathrm{NL}}(|\psi\rangle) = \min_{U_A \otimes U_B} \mathrm{M}(U_A \otimes U_B |\psi\rangle)$$

and its extension to the multipartite case

$$\mathcal{M}^{\mathrm{n-NL}}(|\psi\rangle) = \min_{\otimes_i U_i} \mathrm{M}(\otimes_i U_i |\psi\rangle)$$

CLiCK-P = Cao – Li – Cheng – Karthikeyan – Preskill = Curvature-Like Construction with Kick-responsive Proto-area

Cao C, G. Cheng, A. Hamma, L. Leone, W. Munizzi, and S. FE Oliviero. "Gravitational backreaction is magical." PRX Quantum 6, no. 4 (2025): 040375

Cao, C., Cheng, G., Karthikeyan, K., Li, C., & Preskill, J. (2026). State-dependent geometries from magic-enriched quantum codes. arXiv preprint arXiv:2603.13475.

Biswas, D., et al. (2026). Observation of gravity-like signatures in holographic codes on a quantum computer. arXiv preprint arXiv:2607.12047.